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Quantum Game of Two Discriminable Coins

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Abstract In some recent letters, it was reported that quantum strategies are more successful than classical ones for coin-tossing and roulette game. In this paper, we will solve the quantum game of two discriminable coins. And we develop two methods, analogy method and isolation method, to study this problem.

Keywords Quantum game · Quantum roulette · Two discriminable coins

1 Introduction

Classical game theory [1, 2] has always been applied successfully to economic and industrial decision models to resolve and determine the best possible strategy. Recently, quantum game theory [3–13] has been investigated, which discusses versions of some classical game (for an account of classical game theory, see [14]) where new rules that make explicit use of quantum mechanics lead to new solutions. D.A. Meyer [8] demonstrated that in a classical two-person zero-sum strategic game, if one person adopts a quantum strategy, then he has a better chance of winning the game. And based on these work, Xiang-Bin Wang, L.C. Kwek et al. [15] extended this case by replacing the coin which has only two possible states (namely head and tail) with a roulette with N states, and concluded that quantum strategies can also be more successful than classical ones; Jing-Ling Chen, L.C. Kwek and C.H. Oh [16] studied noisy quantum game. In this paper, we will attempt to solve quantum game problem of two discriminable coins. And we develop two methods, analogy method and isolation method, to study this question.

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2 Single Quantum Coin

Now we firstly review Meyer's strategies. It is well known that a classical coin has only two possible states, namely head and tail. These two states could be represented by the columns:

$$|1\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |0\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1}$$

Classically, there is a choice of 2! possible flips corresponding to all possible permutations of the set {0, 1}. Explicit matrix forms of the two permutation operators are

$$F^0_{N=2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad F^1_{N=2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

After acting on $|0\rangle$ or $|1\rangle$, $F^0_{N=2}$ lets the coin rest in its original state, while $F^1_{N=2}$ transforms the original state into another different state.

Owning to $F^0_{N=2}$ and $F^1_{N=2}$, we can construct a density matrix G_2 as follows:

$$G_2 \equiv \frac{1}{2}(F^0_{N=2} + F^1_{N=2}) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathrm{Tr}\, G_2 = 1. \tag{3}$$

One can easily verify that G_2 commutes with $F^0_{N=2}$ and $F^1_{N=2}$, i.e.,

$$[G_2, F^j_{N=2}] = G_2 F^j_{N=2} - F^j_{N=2} G_2 = 0, \quad (j = 0, 1). \tag{4}$$

Since $F^j_{N=2}$ is unitary, from the above equation we then have an identity

$$G_2 = (1-p)F^0_2 G_2 F^{0\dagger}_2 + pF^1_2 G_2 F^{1\dagger}_2 \tag{5}$$

which is independent upon the parameter p.

The general pure state of a quantum coin is

$$|\chi\rangle = \cos\frac{\theta}{2}|head\rangle + e^{i\phi}\sin\frac{\theta}{2}|tail\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}, \qquad \langle\chi| = \begin{pmatrix} \cos\frac{\theta}{2}, & e^{-i\phi}\sin\frac{\theta}{2} \end{pmatrix} \tag{6}$$

whose corresponding density matrix reads

$$\rho = |\chi\rangle\langle\chi| = \frac{1}{2}\begin{pmatrix} 1+\cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & 1-\cos\theta \end{pmatrix}, \quad \rho^2 = \rho. \tag{7}$$

Now Alice and Bob come to play a coin-tossing game. And Bob could control this game by quantum strategies.

(1) Supposed the initial state of the cion which is placed by Alice is $|\chi_0\rangle$, and its density matrix is $\rho_0 = |\chi_0\rangle\langle\chi_0|$;
(2) Then Bob acts on the coin by a quantum strategy, unitary transformation U_1, instead of a classical stochastic matrix and the state of the coin becomes

$$\rho_1 = U_1 \rho_0 U_1^\dagger = G_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \tag{8}$$

where $U_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ if $\rho_0 = |\chi_0\rangle\langle\chi_0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $U_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ if $\rho_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$;

(3) And Alice continues to using classical tossing. However, from (5) one can know Alice's classical strategy does not change the density matrix of the coin,

$$\rho_2 = pF_2^1\rho_1F_2^{1\dagger} + (1-p)F_2^0\rho_1F_2^{0\dagger} = \rho_1 = G_2; \tag{9}$$

(4) At last, Bob could control the game by an appropriate unitary transformation, $U_2 = U_1^\dagger$. That is, because the density matrix is still G_2 he can adopt an unitary matrix U_2 to transform it into the state which he wants.

3 Two Discriminable Coins

In this section we extend the single quantum coin game to two discriminable quantum coins. Now, we see the first method, namely analogy method. Here we can consider two discriminable coins as a quantum roulette of $N = 4$ after two classical definitions.

We give the first definition as follows:

(1) We define the state that both of coins (denoted by A and B) are head up is the following:

$$A\uparrow B\uparrow = |+\rangle|+\rangle, \tag{10}$$

(2) For the state that A is head up while B is tail up, we define:

$$A\uparrow B\downarrow = |+\rangle|-\rangle, \tag{11}$$

(3) When A is tail up and B is head up, we have

$$A\downarrow B\uparrow = |-\rangle|+\rangle, \tag{12}$$

(4) When both A and B are tail up, we have

$$A\downarrow B\downarrow = |-\rangle|-\rangle. \tag{13}$$

Based on the above definition we can give the second definition so as to solve the problem of two quantum coins by means of quantum roulette of $N = 4$. The reason we can do this is that the numbers of the states of the two quantum coins and the quantum roulette of $N = 4$ are equal.

The second definition is

$$|3\rangle = |+\rangle|+\rangle, \qquad |2\rangle = |+\rangle|-\rangle, \qquad |1\rangle = |-\rangle|+\rangle, \qquad |0\rangle = |-\rangle|-\rangle. \tag{14}$$

From the [14] we can conclude that the states of the roulette of $N = 4$ could be represented by the following matrices

$$|3\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad |2\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \qquad |1\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \qquad |0\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \tag{15}$$

moreover, the states of the two discriminable coins could also be represented the same way under the above definitions. Therefore, we can associate the two coins with the roulette of $N = 4$ in mathematics. The two discriminable coins have four states (that are (14)) which have $4! = 24$ transpositions corresponding to all the permutations of $\{3, 2, 1, 0\}$, and the explicit matrix forms of the 24 permutation operators are

$$F_4^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad F_4^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad F_4^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{16}$$

$$F_4^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad F_4^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad F_4^5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{17}$$

$$F_4^6 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad F_4^7 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad F_4^8 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{18}$$

$$F_4^9 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad F_4^{10} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad F_4^{11} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{19}$$

$$F_4^{12} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad F_4^{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad F_4^{14} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{20}$$

$$F_4^{15} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad F_4^{16} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad F_4^{17} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{21}$$

$$F_4^{18} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad F_4^{19} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad F_4^{20} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{22}$$

$$F_4^{21} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad F_4^{22} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad F_4^{23} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{23}$$

Then we can construct the density matrix G_4,

$$G_4 = \frac{1}{4!} \sum_{j=0}^{23} F_4^j = \frac{1}{24} \cdot 6 \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{24}$$

The same to G_2 of the single quantum coin, G_4 commutes with F_4^j, $(j = 0, 1, \ldots, 23)$, i.e.

$$[G_4, F_4^j] = 0. \tag{25}$$

So we have

$$G_4 = \left(1 - \sum_{j=1}^{23} p_j\right) F_4^0 G_4 F_4^{0+} + \sum_{j=1}^{23} p_j F_4^j G_4 F_4^{j+} \tag{26}$$

which is also independent on parameter p_j.

From

$$\det(G_4 - \lambda I) = \begin{vmatrix} \frac{1}{4} - \lambda & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} - \lambda & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} - \lambda & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} - \lambda \end{vmatrix} = 0, \tag{27}$$

$$-\lambda^3(1 - \lambda) = 0, \tag{28}$$

the eigenvalues of G_4 can be obtained,

$$\lambda_3 = 1, \qquad \lambda_2 = \lambda_1 = \lambda_0 = 0, \tag{29}$$

with the eigenvectors as

$$V_{N=4}^{\lambda_3} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad V_{N=4}^{\lambda_2} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \tag{30}$$

$$V_{N=4}^{\lambda_1} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \qquad V_{N=4}^{\lambda_0} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}. \tag{31}$$

Thus, we can obtain the unitary transformation matrices,

$$S_3 = (V_{N=4}^{\lambda_3}, V_{N=4}^{\lambda_2}, V_{N=4}^{\lambda_1}, V_{N=4}^{\lambda_0}) = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \tag{32}$$

$$S_2 = (V_{N=4}^{\lambda_0}, V_{N=4}^{\lambda_3}, V_{N=4}^{\lambda_2}, V_{N=4}^{\lambda_1}) = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}, \tag{33}$$

$$S_1 = (V_{N=4}^{\lambda_1}, V_{N=4}^{\lambda_0}, V_{N=4}^{\lambda_3}, V_{N=4}^{\lambda_2}) = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}, \tag{34}$$

$$S_0 = (V^{\lambda_2}_{N=4}, V^{\lambda_1}_{N=4}, V^{\lambda_0}_{N=4}, V^{\lambda_3}_{N=4}) = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix} \tag{35}$$

and the diagonalized matrices

$$\Lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Lambda_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{36}$$

$$\Lambda_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Lambda_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{37}$$

with $\Lambda_j = S_j^+ G_4 S_j$ $(j = 0, 1, 2, 3)$, respectively.

Now we consider the two-coin-tossing game and discuss Bob how to control the game by quantum strategies. After the above work we know the two coins game is the same to the roulette of $N = 4$, and we find the diagonalized matrices (36, 37) are the density matrices of the four states of two coins or roulette of $N = 4$, i.e.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longleftrightarrow |3\rangle = |+\rangle|+\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{38}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longleftrightarrow |2\rangle = |+\rangle|-\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \tag{39}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longleftrightarrow |1\rangle = |-\rangle|+\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \tag{40}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longleftrightarrow |0\rangle = |-\rangle|-\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{41}$$

Supposed that the initial state of the two coins (the same to the roulette of $N = 4$) is $|\chi_0\rangle$, and its density matrix is $\rho_0 = |\chi_0\rangle\langle\chi_0|$. Now Alice and Bob come to play a two-coin-tossing game. During the game, Bob adopts a quantum strategy by using a unitary rather than a stochastic matrix to act on the coins, while Alice still adopts the usual classical probabilistic strategy. Firstly, Alice places the two coins on one box and the state of the coins is known by both Alice and Bob. Secondly, Bob uses a unitary transformation U_1 to act on the coins, where, if the initial state is $|3\rangle$, $|2\rangle$, $|1\rangle$ or $|0\rangle$ the unitary transformation U_1 is corresponding S_3, S_2, S_1 or S_0 (see (32–34) and (35)), and then the state of the coins becomes $\rho_1 = U_1 \rho_0 U_1^+$. And thirdly, Alice continues to play by employing a convex sum of unitary

Table 1 The two quantum coins game

Initial state $\|\chi_0\rangle$								Final state	
2 coins	4 roulette	ρ_0	U_1	ρ_1	$\rho_2=\rho_1$	U_2	ρ_3	2 coins	4 roulette
$\|+\rangle\|+\rangle$	$\|3\rangle$	a	S_3	G_4		S_3^+	a	$\|+\rangle\|+\rangle$	$\|3\rangle$
$\|+\rangle\|-\rangle$	$\|2\rangle$	b	S_2	G_4	G_4	S_2^+	b	$\|+\rangle\|-\rangle$	$\|2\rangle$
$\|-\rangle\|+\rangle$	$\|1\rangle$	c	S_1	G_4		S_1^+	c	$\|-\rangle\|+\rangle$	$\|1\rangle$
$\|-\rangle\|-\rangle$	$\|0\rangle$	d	S_0	G_4		S_0^+	d	$\|-\rangle\|-\rangle$	$\|0\rangle$

(deterministic) transformation, namely she perhaps changes the state of the two coins using the transformation F_4^j $(j=0,1,\ldots,23)$ with the probability p_j. Thus, at the end of Alice's turn, the state of the coins is described by the density matrix

$$\rho_2=\left(1-\sum_{j=1}^{23}p_j\right)F_4^0\rho_1F_4^{0\dagger}+\sum_{1=0}^{23}p_jF_4^j\rho_1F_4^{j\dagger}. \tag{42}$$

Finally, Bob transforms the coins using the unitary transformation U_2 so that the density matrix of the final state of the coins is $\rho_3=U_2\rho_2U_2^+$, here if he wants to get the state $|3\rangle$ then $U_2=S_3^\dagger$; and $|2\rangle$, $U_2=S_2^\dagger$; $|1\rangle$, $U_2=S_1^\dagger$; $|0\rangle$, $U_2=S_0^\dagger$. Thus, Bob can control the game by the above quantum strategies. The results are listed in Table 1, where

$$a=\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\qquad b=\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \tag{43}$$

$$c=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix},\qquad d=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}. \tag{44}$$

From Table 1 one can observe that

1. For the initial state $|\chi_0\rangle$, i.e. $|+\rangle|+\rangle=|3\rangle$, $|+\rangle|-\rangle=|2\rangle$, $|-\rangle|+\rangle=|1\rangle$ or $|-\rangle|-\rangle=|0\rangle$, the unitary matrix U_1 (that is corresponding S_3, S_2, S_1, or S_0) could transform the initial density $\rho_0=|\chi_0\rangle\langle\chi_0|$ into $\rho_1=G_4$.
2. When $\rho_1=G_4$, whatever the probability p_j is, ρ_2 is always equal to G_2.
3. Bob can always control the final state of the two coins by using the unitary matrix $U_2=S_3^\dagger$, $S_2^\dagger$, $S_1^\dagger$, or $S_0^\dagger$. Hence Bob can always win the two-coin-tossing game if he wants. For instance, suppose that the initial state of the coins is $|+\rangle|+\rangle$ i.e. the two coins are both head up, and in the end of the game, both Alice and Bob agree that Bob will win the game if the coins are both head up, and will lose otherwise. In this case, Bob could use an appropriate unitary transformation, $U_1=S_3$ $(j=3,2,1,0)$ and $U_2=S_3^\dagger$, which yields $\rho_3=\rho_0$, thus Bob wins. For other initial states Bob could use the similar way to win the game if he wants.

At last, we will use the second method, namely isolation method, to analyze the two quantum coins. Here, because the two coins are discriminable we could consider the two coins as two single coins, and the result is the simple superposition of the two single coins. Thus we can structure the density matrix as follows:

$$G_{2\times 2} = G_2 \otimes G_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{45}$$

Obviously, we obtain the same density matrix as (24), then we could analyze the problem like the first method, and the difference between them is just that we use different methods to obtain the density matrix G_4 or $G_{2\times 2}$.

4 Discussion and Conclusion

To summarize, we introduce two methods, the analogy method and the isolation method, to discuss the quantum game of two discriminable coins. In the first method, we consider the two coins as a roulette which has four states; and in the later, the two coins can be seen as two single coins which are isolating. When we use any method, we can get a matrix G_4, which the density matrix of the initial state of the two coins can be changed to after Bob using a proper unitary transformation U_1, that is $\rho_1 = G_4$. And then, Bob can use another proper unitary transformation U_2 to control the game because this matrix G_4 is invariant under the classical transposition. Therefore, for two discriminable coins, quantum strategies are more successful than classical ones. For any finite discriminable coins (e.g. M coins), we can conjecture that the same conclusion is obtained if only we regard the M coins as a quantum roulette which has $N = 2^M$ states or as M single-coins.

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